<u>The Scattering Matrix</u>

Motivation for introducing the SM:

(1) The open and short circuit required for the Z and Y parameters cannot usually be implemented in actual high-frequency measurements (parasitic C and L);
(2) There may be biasing and/or stability problems for active devices. Hence, it is preferable to measure the two-port under its actual operating conditions;
(3) At microwave frequencies, V and I are hard to measure; only power and phase are detectable.
(4) Always exist.

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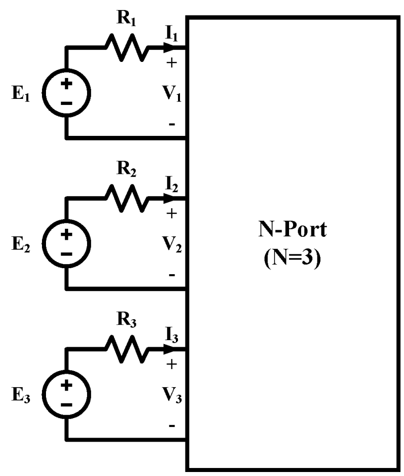
A photograph of the Hewlett-Packard HP8510B Network Analyzer. This test instrument is used to measure the scattering parameters (magnitude and phase) of a one ore two-port microwave network from 0.05 GHz to 26.5 GHz. Built-in microprocessors provide error correction, a high degree of accuracy, and a wide choice of display formats. This analyzer can also perform a fast Fourier transform of the frequency domain data to provide a time domain response of the network under test.

Definitions

We shall use variables, which are directly related to power. The inputs (incident signals) are related to the known maximum available power of the generator at the port; outputs (reflected signals) are related to the actual output power at the port.

Assume resistive terminations R_k at all ports, with or without a generator E_k. The actual port variables are V_k and I_k, considered peak value phasors of steady-state sine-wave signals.



We define the following hypothetical quantities:

Incident voltage V_{ik} at port k: the output voltage $E_k/2$ of the generator at port k under matching (max-power) conditions.

Incident current I_{ik} at port k: the output current $Ek/(2R_k)$ of the generator at port k under matching conditions.

Incident wave signal a_k at port k: $a_k = \sqrt{V_{ik} I_{ik}} = \frac{V_{ik}}{\sqrt{R_k}} = I_{ik}\sqrt{R_k}$

Hence, $\frac{|a_k|^2}{2} = \frac{E_k^{\ 2}}{8R_k} = P_{\max k}$

<u>Reflected voltage</u> V_{rk} at port k: the difference between the actual and incident voltages;

$V_{rk} = V_k - V_{ik}$. If there is no generator, V_{rk} equals the actual voltage at port k.

<u>Reflected current</u> I_{rk} at port k: The difference between the incident and actual currents (notice the sign difference!); $I_{RK} = I_{ik} - I_k$. If there is no generator at port k, $I_{rk} = -I_k$. $V_{rk} = R_k I_{rk}$.

<u>Reflected wave</u> (signal) b_k at port k: $b_k = \sqrt{V_{rk} I_{rk}} = V_{rk} / \sqrt{R_k} = I_{rk}\sqrt{R_k}$

Hence $| b_k |^2 / 2 = V_{rk} \cdot I_{rk} / 2$. If there is no generator, this is the actual power exiting at port k.

As will be shown, the actual power entering at pork k where there is a generator is

$\left[| a_k |^2 - | b_k |^2\right]/2.$

<u>Scattering Matrix</u> <u>S</u> connects the column vectors <u>a</u> and <u>b</u> formed from all incident and reflected signals: <u>b</u> = <u>S</u> <u>a</u>

From this definition it follows that the lm element of <u>S</u> is given by

$$S_{lm} = \frac{b_l}{a_m} = \frac{V_{r_l} / \sqrt{R_l}}{E_m / \left(2\sqrt{R_m}\right)} = \frac{V_l / \sqrt{R_l}}{E / \left(2\sqrt{R_m}\right)}, \quad l \neq m$$

Where $E_k = 0$ is set for all $k \neq m$.

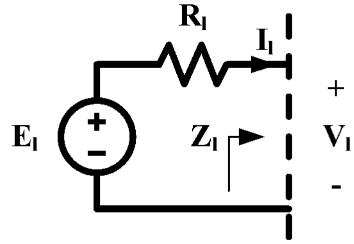
$| S_{lm} |^2$ has a clear physical meaning:

$$| S_{lm} |^2 = \frac{P_{rl}}{P_{\max m}} = \frac{\text{actual power leaving port } l}{\text{maximum power from port } m}$$

$E_k = 0$ for $k \neq m$.

For the diagonal element s_{ll}, with the only generator E_l.

$$V_l = E_l \frac{Z_l}{Z_l + R_l}$$



Which gives

$$\frac{b_l}{a_l} = s_{ll} = \frac{Z_l - R_l}{Z_l + R_l}$$

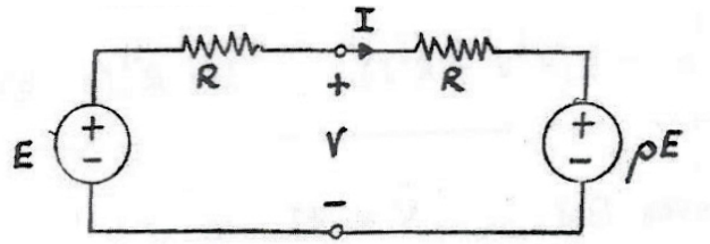
Reflection Coefficient at port l:

Gives the ratio of the reflected and incident powers. The power entering port l is therefore given by:

$$P_l = P_{il} - P_{rl} = P_{il}\left[1 - |S_{ll}|^2\right]$$

Noted that the scattering relations given above describe not only the N-port (like e.g., the open-circuit impedance equations) but also its terminations and sources.

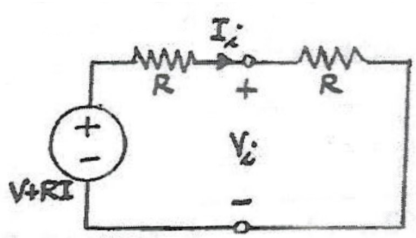
Scattering variables

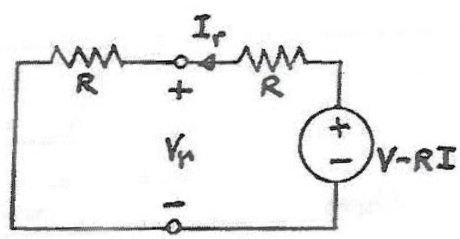


By KVL, E= V+RI ρE = V-RI = $\frac{V-RI}{V+RI}E$

So ρ=$\frac{V-RI}{V+RI}=\frac{Z-R}{Z+R}$ the Reflectance (Reflection Coefficient)

By superposition:





$$V_i=\frac{1}{2}(V+RI) \qquad V_r=\frac{1}{2}(V-RI)$$
$$I_i=\frac{1}{2R}(V+RI) \qquad I_r=\frac{1}{2R}(V-RI)$$
$$V_i = RI_i \qquad V_r = RI_r$$

We define:

Incident signal = a = $\sqrt{V_iI_i}$
$= R^{-\frac{1}{2}}V_i$
$= R^{\frac{1}{2}}I_i$

Reflected signal = b =$\sqrt{V_rI_r}$
$= R^{-\frac{1}{2}}V_r$
$= R^{\frac{1}{2}}I_r$

$$a = \frac{1}{2}\left(R^{-\frac{1}{2}}V + R^{\frac{1}{2}}I\right) \qquad V = R^{\frac{1}{2}}(a+b)$$

$$b = \frac{1}{2}\left(R^{-\frac{1}{2}}V - R^{\frac{1}{2}}I\right) \qquad I = R^{-\frac{1}{2}}(a-b)$$

Given that V = ZI

$$R^{\frac{1}{2}}(a+b) = ZR^{-\frac{1}{2}}(a-b)$$

$$a + b = z_n(a-b) \qquad \text{where } Z_n \equiv R^{-\frac{1}{2}}ZR^{-\frac{1}{2}}$$

$$(Z_n+1)b = (Z_n-1)a$$

$$b = (Z_n+1)^{-1}(Z_n-1)a$$

$$b = Sa$$

where $S = (Z_n+1)^{-1}(Z_n-1) \qquad = \rho = (Z-R)/(Z+R)$

If the 1-port is passive, Z & Z_n are positive functions.

i.e. $\mathrm{Re}\{Z\} \geq 0$ for $\sigma \geq 0$ & the same for Z_n

Then S satisfies $|S| \leq 0$ for $\sigma \geq 0$

S is then called a <u>bounded function</u>

Z_n is positive iff S is bounded.

Power relations

The active power absorbed by the 1-port is

$$W_a = \frac{1}{2}\mathrm{Re}\left\{\bar{I}Ve^{2\sigma t}\right\} = \frac{1}{2}\mathrm{Re}\left\{(\bar{a}-\bar{b})R^{-\frac{1}{2}}R^{\frac{1}{2}}(a+b)e^{2\sigma t}\right\}$$

$$= \frac{1}{2}e^{2\sigma t}\,\mathrm{Re}\left\{(\bar{a}-\bar{b})(a+b)\right\}$$

$$= \frac{1}{2}e^{2\sigma t}\,\mathrm{Re}\left\{(\bar{a}a-\bar{b}b)+(\bar{a}b-\bar{b}a)\right\}$$

Real Imag.

$$W_a = \frac{1}{2}e^{2\sigma t}\left\{|a|^2 - |b|^2\right\}$$

Since b =Sa & $|b|^2 = |S|^2\,|a|^2$

$$\underline{W_a = \frac{1}{2}e^{2\sigma t}(1-|S|^2)\,|a|^2}$$

In the harmonic state $\sigma = 0$ & $W_a = P$

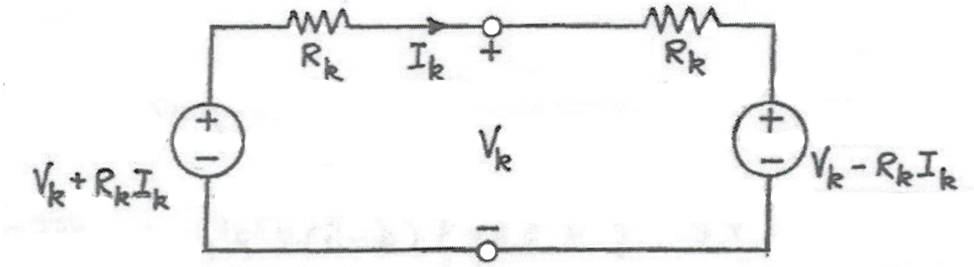
$$P = \frac{1}{2}(1-|S|^2)\,|a|^2$$

P is a maximum when $|S|^2 = 0$ i.e. Z = R

So $P_{max} = \frac{1}{2}|a|^2$ is maximum available power from source.

Hence $\underline{P = (1-|S|^2)\,P_{max}}$

N-port case



$\underline{V}$ = n-vector of port voltages

$\underline{I}$ = n-vector of port current

$\underline{R}$ = diag (R_1, R_2, …,R_n) & $\underline{R}^{-1}$

$\underline{R}^{\frac{1}{2}}$ = diag ($R_1^{\frac{1}{2}}, R_2^{\frac{1}{2}}, \ldots, R_n^{\frac{1}{2}}$) & $\underline{R}^{-\frac{1}{2}}$

Exactly as in the 1-port case

$$\underline{a} = \frac{1}{2}\left(\underline{R}^{-\frac{1}{2}}\underline{V} + \underline{R}^{\frac{1}{2}}\underline{I}\right) \qquad \underline{V} = \underline{R}^{\frac{1}{2}}(\underline{a} + \underline{b})$$

$$\underline{b} = \frac{1}{2}\left(\underline{R}^{-\frac{1}{2}}\underline{V} - \underline{R}^{\frac{1}{2}}\underline{I}\right) \qquad \underline{I} = \underline{R}^{-\frac{1}{2}}(\underline{a} - \underline{b})$$

$$\underline{V_i} = \underline{R}^{\frac{1}{2}}\underline{a} \qquad \underline{I_i} = \underline{R}^{-\frac{1}{2}}\underline{a}$$

$$\underline{V_r} = \underline{R}^{\frac{1}{2}}\underline{b} \qquad \underline{I_r} = \underline{R}^{-\frac{1}{2}}\underline{b}$$

The scattering matrix is defined by

$\underline{b} = \underline{S}\ \underline{a}$

<u>Relation between $\underline{S}$ and $\underline{Z}$</u>

$$\underline{V} = \underline{Z}\underline{I}$$

$$\underline{R}^{\frac{1}{2}}(a+b) = \underline{Z}\underline{R}^{-\frac{1}{2}}(\underline{a}-\underline{b})$$

$$\underline{a}+\underline{b} = \underline{R}^{-\frac{1}{2}}\underline{Z}\underline{R}^{-\frac{1}{2}}(\underline{a}-\underline{b}) = \underline{Z_n}(\underline{a}-\underline{b}) \qquad \underline{Z_n} = \underline{R}^{-\frac{1}{2}}\underline{Z}\underline{R}^{-\frac{1}{2}}$$

$$(\underline{Z_n}+\underline{1})\underline{b} = (\underline{Z_n}-\underline{1})\underline{a}$$

$$\underline{b} = (\underline{Z_n}+\underline{1})^{-1}(\underline{Z_n}-\underline{1})\underline{a}$$

$$\boxed{\underline{S} = (\underline{Z_n}+\underline{1})^{-1}(\underline{Z_n}-\underline{1}) = (\underline{Z_n}-\underline{1})(\underline{Z_n}+\underline{1})^{-1}}$$

$$\begin{aligned}\underline{S} &= (\underline{Z_n}-\underline{1})(\underline{Z_n}+\underline{1})^{-1} = (\underline{Z_n}+\underline{1}-\underline{2})(\underline{Z_n}+\underline{1})^{-1} \\ &= \underline{1}-2(\underline{Z_n}+\underline{1})^{-1} = \underline{\underline{\underline{1}-2\underline{Y_{an}}}}\end{aligned}$$

Where

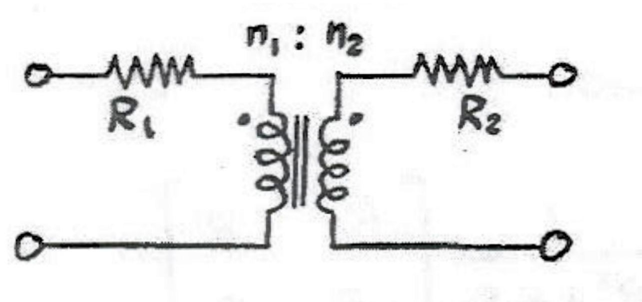
$$\begin{aligned}\underline{Y_{an}} &= (\underline{Z_n}+\underline{1})^{-1} = (\underline{R}^{-\frac{1}{2}}\underline{Z}\underline{R}^{-\frac{1}{2}} + \underline{R}^{-\frac{1}{2}}\underline{R}\underline{R}^{-\frac{1}{2}})^{-1} \\ &= \left[\underline{R}^{-\frac{1}{2}}(\underline{Z}+\underline{R})\underline{R}^{-\frac{1}{2}}\right]^{-1} \\ &= \underline{R}^{\frac{1}{2}}(\underline{Z}+\underline{R})^{-1}\underline{R}^{\frac{1}{2}}\end{aligned}$$

(Normalized & augmented $\underline{Y}$ matrix)

As $\underline{Y_a}$, and hence $\underline{Y_{an}}$, always exists for any well-defined n-port, it means that $\underline{S}$ also always exists.

Examples

Ideal transformer



$$\underline{Y_a} = \frac{1}{n_2^2 R_1 + n_1^2 R_2}\begin{bmatrix} n_2^2 & -n_1 n_2 \\ -n_1 n_2 & n_1^2 \end{bmatrix}$$

$$\underline{Y_{an}} = \frac{1}{n_2^2 R_1 + n_1^2 R_2}\begin{bmatrix} n_2^2 R_1 & -n_1 n_2 \sqrt{R_1 R_2} \\ -n_1 n_2 \sqrt{R_1 R_2} & n_1^2 R_2 \end{bmatrix}$$

$$\underline{S} = \underline{1} - 2\underline{Y_{an}} = \frac{1}{n_2^2 R_1 + n_1^2 R_2}\begin{bmatrix} n_1^2 R_2 - n_2^2 R_1 & 2 n_1 n_2 \sqrt{R_1 R_2} \\ 2 n_1 n_2 \sqrt{R_1 R_2} & n_2^2 R_1 - n_1^2 R_2 \end{bmatrix}$$

If $\frac{R_1}{R_2} = \frac{n_1^2}{n_2^2}$ or $n_1^2 R_2 = n_2^2 R_1$

Then $\underline{S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Alternative method

Alternative (ideal transformer)

$$S_{11} = \frac{Z_1 - R_1}{Z_1 + R_1} = \frac{(n_1/n_2)^2 R_2 - R_1}{(\quad)^2 R_2 + R_1} = \frac{n_1^2 R_2 - n_2^2 R_1}{n_1^2 R_2 + n_2^2 R_1}$$

$$a_1 = E_1 / 2\sqrt{R_1}$$

$$b_2 = V_2/\sqrt{R_2} = \frac{n_2}{n_1} \frac{V_1}{\sqrt{R_2}} = \frac{n_2}{n_1} E_1 \frac{(n_1/n_2)^2 R_2}{R_1 + (n_1/n_2)^2 R_2} \frac{1}{\sqrt{R_2}}$$

$$= \frac{n_2 E_1}{n_1 \sqrt{R_2}} \frac{n_1^2 R_2}{n_1^2 R_2 + n_2^2 R_1}$$

$$S_{21} = \frac{b_2}{a_1} = 2\sqrt{\frac{R_1}{R_2}} \frac{n_2}{n_1} \frac{n_1^2 R_2}{n_1^2 R_2 + n_2^2 R_1} = \frac{2 n_1 n_2 \sqrt{R_1 R_2}}{n_1^2 R_2 + n_2^2 R_1}$$

Going back to

$$\underline{S} = (\underline{Z_n} + 1)^{-1}(\underline{Z_n}^{-1} - \underline{1})$$

$$= \left[\underline{R^{-\frac{1}{2}}}\underline{Z}\underline{R^{-\frac{1}{2}}} + \underline{R^{-\frac{1}{2}}}\underline{R}\underline{R^{-\frac{1}{2}}}\right]^{-1}\left[\underline{R^{-\frac{1}{2}}}\underline{Z}\underline{R^{-\frac{1}{2}}} - \underline{R^{-\frac{1}{2}}}\underline{R}\underline{R^{-\frac{1}{2}}}\right]$$

$$= \left[\underline{R^{-\frac{1}{2}}}(\underline{Z} + \underline{R})\underline{R^{-\frac{1}{2}}}\right]^{-1}\left[\underline{R^{-\frac{1}{2}}}(\underline{Z} - \underline{R})\underline{R^{-\frac{1}{2}}}\right]$$

$$= \underline{R^{\frac{1}{2}}}(\underline{Z} + \underline{R})^{-1}\underline{R^{\frac{1}{2}}}\underline{R^{-\frac{1}{2}}}(\underline{Z} - \underline{R})\underline{R^{-\frac{1}{2}}}$$

$$\underline{S} = \underline{R^{\frac{1}{2}}}(\underline{Z} + \underline{R})^{-1}(\underline{Z} - \underline{R})\underline{R^{-\frac{1}{2}}}$$

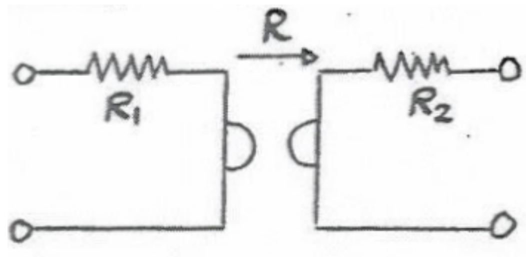
<u>Normalization</u>

$$\underline{R^{-\frac{1}{2}}}\underline{Z}\underline{R^{-\frac{1}{2}}} = \underline{Z_n}$$

Divides row i by $\sqrt{R_i}$ Divides col j by $\sqrt{R_j}$

(i, j)th element of $\underline{Z_n}$ is $\dfrac{Z_{ij}}{\sqrt{R_i R_j}}$

<u>Gyrator</u>



$$\underline{Y_a} = \begin{bmatrix} R_1 & -R \\ R & R_2 \end{bmatrix}^{-1} = \frac{1}{R^2 + R_1R_2}\begin{bmatrix} R_2 & R \\ -R & R_1 \end{bmatrix}$$

$$\underline{Y_{an}} = \frac{1}{R^2 + R_1R_2}\begin{bmatrix} R_1R_2 & R\sqrt{R_1R_2} \\ -R\sqrt{R_1R_2} & R_1R_2 \end{bmatrix}$$

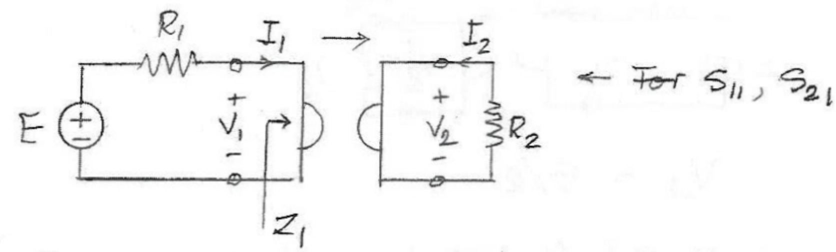
$$\underline{S} = \underline{1} - 2\underline{Y_{an}} = \frac{1}{R^2 + R_1R_2}\begin{bmatrix} R^2 - R_1R_2 & -2R\sqrt{R_1R_2} \\ 2R\sqrt{R_1R_2} & R^2 - R_1R_2 \end{bmatrix}$$

If $R_1R_2 = R^2$

Then $\underline{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

<u>Note:</u> We do not require $R_1=R_2=R$

Direct calculation:



$$Z_1 = \frac{R^2}{R_2}$$

$$V_1 = E\frac{Z_1}{Z_1 + R_1} = \frac{ER^2}{R^2 + R_1R_2}, \qquad V_{i1} \equiv \frac{E}{2}, \qquad a_1 = \frac{E}{2\sqrt{R_1}}$$

$$b_1 = V_{r1}/\sqrt{R_1} = (V_1 - V_{i1})/\sqrt{R_1}$$

$$S_{11} = \frac{b_1}{a_1} = \frac{R^2 - R_1R_2}{R^2 + R_1R_2}$$

$$V_2 = V_{r2} = RI_1 = R\frac{E}{R_1 + Z_1}$$

$$b_2 = \frac{V_{r2}}{\sqrt{R_2}} = \frac{E}{\sqrt{R_2}}\frac{R}{R_1 + Z_1} = E\frac{R\sqrt{R_2}}{R_1R_2 + R^2}$$

$$S_{21} = \frac{b_2}{a_1} = \frac{2R\sqrt{R_1R_2}}{R_1R_2 + R^2}$$

To find S_{12} & S_{22}, move E is series with R_2

Then, $R_1 \leftrightarrow R_2$, $R \rightarrow -R$, hence $S_{22} = S_{11}$, $S_{12} = -S_{21}$

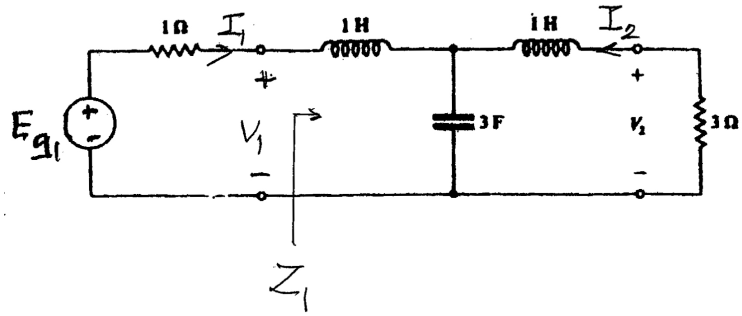
Pozar

	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\dfrac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\dfrac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\dfrac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\dfrac{2Z_{12}Z_0}{\Delta Z}$	$\dfrac{-2Y_{12}Y_0}{\Delta Y}$	$\dfrac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\dfrac{2Z_{21}Z_0}{\Delta Z}$	$\dfrac{-2Y_{21}Y_0}{\Delta Y}$	$\dfrac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\dfrac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\dfrac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\dfrac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\dfrac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\dfrac{Y_{22}}{\|Y\|}$	$\dfrac{A}{C}$
Z_{12}	$Z_0\dfrac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\dfrac{-Y_{12}}{\|Y\|}$	$\dfrac{AD-BC}{C}$
Z_{21}	$Z_0\dfrac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\dfrac{-Y_{21}}{\|Y\|}$	$\dfrac{1}{C}$
Z_{22}	$Z_0\dfrac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\dfrac{Y_{11}}{\|Y\|}$	$\dfrac{D}{C}$
Y_{11}	$Y_0\dfrac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\dfrac{Z_{22}}{\|Z\|}$	Y_{11}	$\dfrac{D}{B}$
Y_{12}	$Y_0\dfrac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\dfrac{-Z_{12}}{\|Z\|}$	Y_{12}	$\dfrac{BC-AD}{B}$
Y_{21}	$Y_0\dfrac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\dfrac{-Z_{21}}{\|Z\|}$	Y_{21}	$\dfrac{-1}{B}$
Y_{22}	$Y_0\dfrac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\dfrac{Z_{11}}{\|Z\|}$	Y_{22}	$\dfrac{A}{B}$
A	$\dfrac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\dfrac{Z_{11}}{Z_{21}}$	$\dfrac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\dfrac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\dfrac{\|Z\|}{Z_{21}}$	$\dfrac{-1}{Y_{21}}$	B
C	$\dfrac{1}{Z_0}\dfrac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\dfrac{1}{Z_{21}}$	$\dfrac{-\|Y\|}{Y_{21}}$	C
D	$\dfrac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\dfrac{Z_{22}}{Z_{21}}$	$\dfrac{-Y_{11}}{Y_{21}}$	D

$|Z| = Z_{11}Z_{22} - Z_{12}Z_{21}$; $|Y| = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$

Find the scattering matrix S(s) of the circuit shown using direct analysis.

(Note: the circuit is the same as on p.74 of the notes.)



From the circuit,

$$Z_1 = \frac{3s^3 + 9s^2 + 2s + 3}{3s^2 + 9s + 1}$$

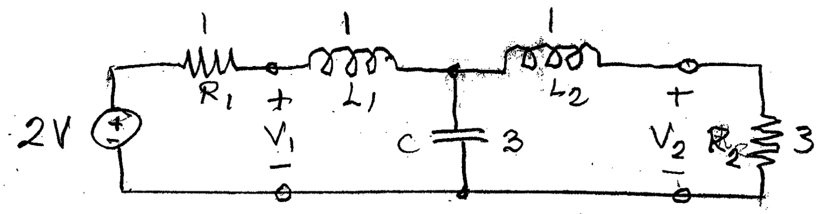
$$S_{11} = \frac{Z_1 - R_1}{Z_1 + R_1} = \frac{3s^3 + 6s^2 - 7s + 2}{3s^3 + 12s^2 + 11s + 4}$$

By circuit analysis,

$$V_2 = \frac{3E_{g1}}{3s^3 + 12s^2 + 11s + 4} = V_{2r}$$

$$S_{21} = 2\sqrt{\frac{R_1}{R_2}}\frac{V_{2r}}{E_{g1}} = \frac{2\sqrt{3}}{3s^3 + 12s^2 + 11s + 4}$$

Method of using $\underline{Y_{an}}$



$\omega = 1$ ra/s

$$\underset{\sim}{Z_a} = \begin{bmatrix} R_1 + sL_1 + 1/sC & 1/sC \\ 1/sC & R_2 + sL_2 + 1/sC \end{bmatrix}$$

$$= \begin{bmatrix} 1 + s + \frac{1}{3s} & \frac{1}{3s} \\ \frac{1}{3s} & 3 + s + \frac{1}{3s} \end{bmatrix}$$

$$\underset{\sim}{Y_a} = \frac{1}{3s^3 + 12s^2 + 11s + 4} \begin{bmatrix} 3s^2 + 9s + 1 & -1 \\ -1 & 3s^2 + 3s + 1 \end{bmatrix}$$

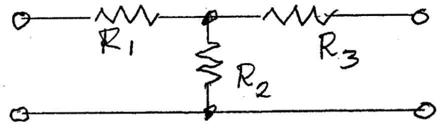
$$\underset{\sim}{Y_{an}} = \frac{1}{3s^3 + 12s^2 + 11s + 4} \begin{bmatrix} 3s^2 + 9s + 1 & -\sqrt{3} \\ -\sqrt{3} & 9s^2 + 9s + 3 \end{bmatrix}$$

$$\underset{\sim}{S} = \underset{\sim}{\mathbb{1}} - 2\underset{\sim}{Y_{an}} = \frac{1}{3s^3 + 12s^2 + 11s + 4} \begin{bmatrix} 3s^3 + 6s^2 - 7s + 2 & 2\sqrt{3} \\ 2\sqrt{3} & 3s^3 - 6s^2 - 7s - 2 \end{bmatrix}$$

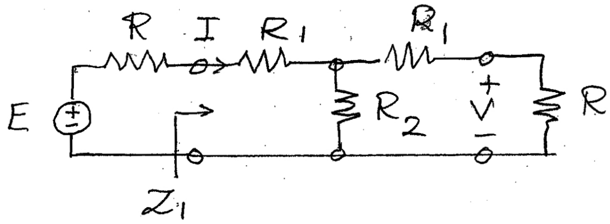
1. The twoport shown operates between two 50-ohm terminations. Its scattering parameters must be S11 = S22 = 0, and S12 = S21 = 0.707.

a. Find the element values R1, R2 and R3.

b. What function can the twoport perform?



1. Symmetric terminations and S_{ij}, hence $R_3 = R_1$



$$Z_1 = R_1 + \frac{R_2(R_1+R)}{R_1+R_2+R}$$

$$S_{11} = 0 \rightarrow Z_1 = R$$

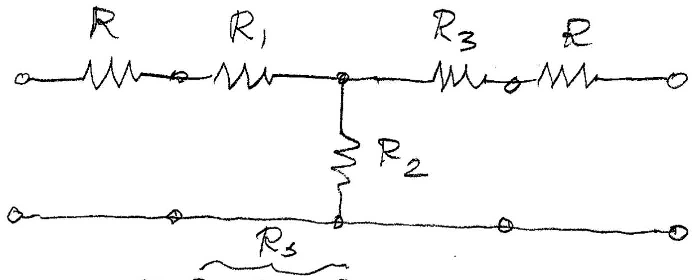
$$S_{21} = \frac{1}{\sqrt{2}} = 2\frac{V}{E}, \quad I = \frac{E}{R+Z_1} = \frac{E}{2R}, \quad V = I\frac{R_2 R}{R_1+R_2+R}$$

$$\frac{1}{\sqrt{2}} = \frac{2}{E}\frac{E}{2R}\frac{R_2 R}{R_1+R_2+R} = \frac{R-R_1}{R+R_1} \rightarrow (1+\sqrt{2})R_1 = (\sqrt{2}-1)R$$

$R_1 = R_3 \cong 8.58\,\Omega$, $R_2 \cong 141.4\,\Omega$

3-dB attenuator.

Method of using $\underline{Y_{an}}$



$$\underset{\sim}{Z_a} = \begin{bmatrix} R + R_1 + R_2 & R_2 \\ R_2 & R_s \end{bmatrix}$$

$$\underset{\sim}{Y_a} = \begin{bmatrix} R_s & -R_2 \\ -R_2 & R_s \end{bmatrix} \frac{1}{R_s^2 - R_2^2}$$

$$\underset{\sim}{Y_{an}} = \frac{R}{R_s^2 - R_2^2} \begin{bmatrix} R_s & -R_2 \\ -R_2 & R_s \end{bmatrix}$$

$$\underset{\sim}{S} = \underset{\sim}{1} - 2\underset{\sim}{Y_{an}} = \frac{1}{R_s^2 - R_2^2} \begin{bmatrix} R_s^2 - R_2^2 - 2R_sR & 2RR_2 \\ 2RR_2 & \longrightarrow \end{bmatrix}$$

For $S_{11} = S_{22} = 0$, $\quad R_s^2 - R_2^2 - 2R_sR = 0$

For $S_{12} = S_{21} = \frac{1}{\sqrt{2}}$, $\quad R_s^2 - R_2^2 - 2\sqrt{2}R_2R = 0$

$$R_s = \sqrt{2} \cdot R_2$$

$$2R_2^2 - R_2^2 - 2\sqrt{2}R_2 R = 0$$

$$R_2 = 2\sqrt{2}R \cong 141.4\,\Omega$$

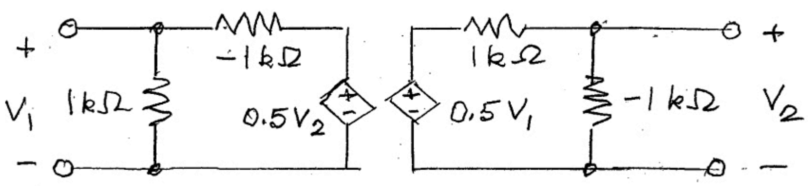
$$R_s = 4R = 200\,\Omega$$

$$R_1 = R_s - R - R_2 \cong 8.58\,\Omega$$

2.a. Find the $\underset{\sim}{Y}$ matrix of the twoport shown.

b. Find its scattering matrix if it is connected between terminations R1 = 1 kΩ and R2 = 4 kΩ.

c. What function does the twoport perform?



2. $$\underset{\sim}{I} = \begin{bmatrix} G_a + G_b & -k_2 G_b \\ -k_1 G_c & G_c + G_d \end{bmatrix} \underset{\sim}{V}$$

So

$$\underset{\sim}{Y} = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix},$$ where $g = 0.5\,mS$

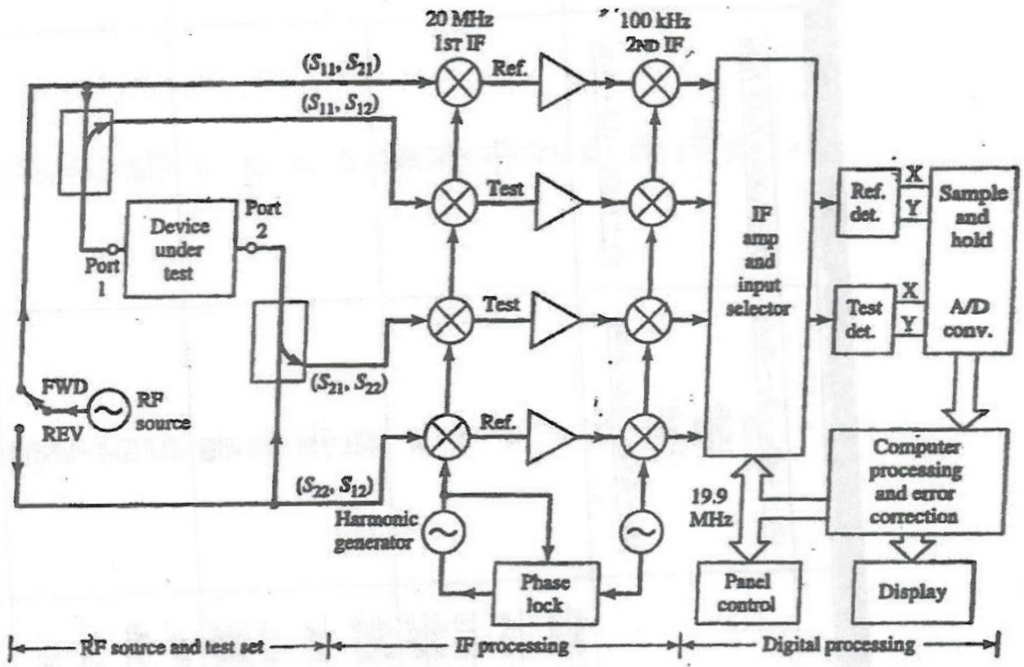
So the circuit is a gyrator, with $r = 2\,k\Omega$. Since $R_1 R_2 = 4\,k\Omega^2 = r^2$,

$$\underset{\sim}{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Gyrator

POINT OF INTEREST: The Vector Network Analyzer

The S parameters of passive and active networks can be measured with a vector network analyzer, which is a two- (or four-) channel microwave receiver designed to process the magnitude and phase of the transmitted and reflected waves from the network. A simplified block diagram of a network analyzer similar to the HP8510 system is shown below. In operation, the RF source is usually set to sweep over a specified bandwidth. A four-port reflectometer samples the incident, reflected, and transmitted RF waves; a switch allows the network to be driven from either port 1 or port 2. Four dual-conversion channels convert these signals to 100 kHz IF frequencies, which are then detected and converted to digital form. A powerful internal computer is used to calculate



and display the magnitude and phase of the S parameters, or other quantities that can be derived from the S parameters, such as SWR, return loss, group delay, impedance, etc. An important feature of this network analyzer is the substantial improvement in accuracy made possible with error correcting software. Errors caused by directional coupler mismatch, imperfect directivity, loss, and variations in the frequency response of the analyzer system are accounted for by using a twelve-term error model and a calibration procedure. Another useful feature is the capability to determine the time domain response of the network by calculating the inverse Fourier transform of the frequency domain data.